JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **37**, No. 3, August 2024
http://dx.doi.org/10.14403/jcms.2024.37.3.117

SOME CONDITIONS ON THE EXISTENCE OF THE PAIRS OF PRIMES THAT DIFFER BY $2k$

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Abstract. Let p be a prime number. The pair $(p, p+2)$ is called a twin prime pair when $p+2$ is also a prime number. In this paper, we find a equation which makes two propositions, integers that are not able to expressed into the equation are infinite, and twin prime pair are infinite, are equivalent. Especially, we find the form of n which makes $(an-k, an+k)$ is prime pair for an integer a and an odd number k with a difference $2k$.

1. Introduction

A twin prime pair $(p, p+2)$ is a pair which two numbers p and $p+2$ are prime numbers. Whether the twin prime pairs are infinite or not is still open problem, which is researched actively. Famous mathematicians conjectured that twin prime pairs and the generalized version are infinite, but that are not proved until now. Maria Suzuki [1] finds that there are infinitely many twin prime pairs if and only if there are infinitely many positive integers n that cannot be written in the form

$$n = 6|ab| + a + b$$

for all positive integers a and b.

We can generalize twin prime pair as the difference from 2 to $2k$, and it is called a prime pair with difference $2k$. In this paper, we find the equation which makes two proposition, integers that are not able to expressed into the equation are infinite and prime pairs with difference $2k$ are infinite, are equivalent.

Received March 04, 2024; Accepted August 20, 2024.
2020 Mathematics Subject Classification: Primary 11A41, 11N080 ; Secondary 11N05.
Key words and phrases: Generalized Twin Prime.
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2. Results

Using the idea about the twin prime problem, we find about the general twin prime with the difference of other even number.

LEMMA 2.1. *All the general twin prime pair with difference* 4 *can be expressed in the form of* $(3n-2, 3n+2)$.

Proof. All the prime numbers except 3 should not be the multiple of 3, which means if we divide it, the remainder is 1 or 2. If we set the remainder of p is 2 in the form of $(p, p+4)$, the remainder of $p+4$ is 0, which means it is not a prime. Therefore, we could know that the remainder of p is 1, which means if we set p as a form of $3n-2$, we could get the form, $(p, p+4) = (3n-2, 3n+2)$. □

THEOREM 2.2. *It is equivalent that there are infinitely many integers* n *that cannot be expressed as* $3|ab| + a + 2b$ *where* a *and* b *are integers that* $a \neq 0, \pm 1$ *and* $b \neq 0$, *and there are infinitely many general twin prime pairs with the difference 4 and having the form* $(3n-2, 3n+2)$.

Proof. Let us prove first that if there are infinitely many integers n which cannot be expressed in the form

$$3|ab| + a + 2b$$

are infinite then there are infinitely many twin prime pairs $(3n-2, 3n+2)$. If n cannot be expressed in that form then n should be a form of

$$(3a+2)(3b+1), (3a+2)(3b-1), (3a-2)(3b+1) \text{ or } (3a-2)(3b-1)$$

where a, b are positive integers, since $3n+2$ and $3n-2$ has the remainder 2 and 1 for 3, respectively. Let us find the n can be expressed in the form in all cases.

1. $3n+2 = (3a+2)(3b+1) \Leftrightarrow n = 3|ab| + a + 2b$
2. $3n+2 = (3a-2)(3b-1) \Leftrightarrow n = 3|(-a)(-b)| + (-a) + 2(-b)$
3. $3n-2 = (3a+2)(3b-1) \Leftrightarrow n = 3|a(-b)| + a + 2(-b)$
4. $3n-2 = (3a-2)(3b+1) \Leftrightarrow n = 3|(-a)b| + (-a) + 2b$

This means if the integer n can be expressed in the form $3|ab| + a + 2b$ then at least one of integers $3n-2$ and $3n+2$ is a composite number. Since there is no other way to express a composite number with remainder 1 or 2 for 3, if there is an integer n that could not be expressed in the form then $3n-2$ and $3n+2$ are prime numbers. Therefore, if there are infinitely many n that cannot be expressed in the form then there are infinitely many twin prime pairs $(3n-2, 3n+2)$.

Next, let us prove that if there are infinitely many twin prime pairs with difference 4 then there are infinitely many integers n that cannot be expressed in the form $3\,|ab| + a + 2b$. By the Lemma 2.1, all twin prime pairs with difference 4 except $(3, 7)$ should be a form of $(3n-2, 3n+2)$. Suppose that the integer n can be expressed in the form of $3\,|ab|+a+2b$ then there are four cases.

1. The case of $a > 0$ and $b > 0$:
   Since $3n + 2 = 3(3ab + a + 2b) + 2 = (3a + 2)(3b + 1)$, the number $3n + 2$ is a composite number, which is contradiction.
2. The case of $a > 0$ and $b < 0$:
   Since $3n - 2 = 3(3a(-b) + a + 2b) - 2 = (3a - 2)(3(-b) + 1)$, the number $3n - 2$ is a composite number, which is contradiction.
3. The case of $a < 0$ and $b > 0$:
   $3n - 2 = 3(3(-a)b + a + 2b) - 2 = (3(-a) + 2)(3b - 1)$, the number $3n - 2$ is a composite number, which is contradiction.
4. The case of $a < 0$ and $b < 0$:
   Since $3n+2 = 3(3(-a)(-b)+2+2b)+2 = (3(-a)-2)(3(-b)-1)$, the number $3n + 2$ is a composite number, which is contradiction.

Hence, n cannot be the expressed in the form of $3\,|ab|+a+2b$. Therefore, if there are infinitely many twin prime pairs with the difference 4 then there are infinitely many integers n that cannot be expressed in the form $3\,|ab| + a + 2b$.

□

Remark 2.3. We do not consider the cases that $a = \pm 1$, since $3a - 2$ and $3a + 2$ is 1 and -1, respectively, for each cases. Hence, it cannot be guaranteed that $(3a \pm 2)(3b \pm 1)$ is a composite number.

Similarly like the proof of the theorem 2.2, we find that if n has the form of $n = (p+1)\,|ab| + a + pb$ then at least one of integers $(p+1)n - p$ and $(p + 1)n + p$ is a composite number. Also, if at least one of integers $(p + 1)n - p$ and $(p + 1)n + p$ is a composite number then the integer n can be expressed in the form $(p + 1)\,|ab| + a + pb$ where a and b are integers that $a \neq 0, \pm 1$ and $b \neq 0$. Therefore, we have the following theorem which can be considered as a generalization of the theorem 2.2.

Theorem 2.4. *Let p be a prime number. Then it is equivalent that there are infinitely many integers n that cannot be expressed as $(p + 1)\,|ab| + a + pb$ where a and b are integers that $a \neq 0, \pm 1$ and $b \neq 0$, and there are infinitely many general twin prime pairs with the difference $2p$ and having the form $((p + 1)n - p, (p + 1)n + p)$.*

Let us check the infinity of different types of prime pairs.

THEOREM 2.5. *It is equivalent that there are infinitely many integers n that cannot be expressed as $2\,|ab| + a + 3b$ where a and b are integers that $a \neq 0, \pm 2$ and $b \neq 0$, and there are infinitely many general twin prime pairs with the difference 6 and having the form $(2n-3, 2n+3)$.*

Proof. Let us consider the infinity of twin prime pairs $(2n-3, 2n+3)$ where $n \geq 2$ is an integer. If at least one of integers $2n-3$ and $2n+3$ is a composite number then the following four cases are possible.

1. $2n+3 = (2a+3)(2b+1)$, $n = 2|ab| + a + 3b$
2. $2n+3 = (2a-3)(2b-1)$, $n = 2|(-a)(-b)| + (-a) + 3(-b)$
3. $2n-3 = (2a-3)(2b+1)$, $n = 2|a(-b)| + a + 3(-b)$
4. $2n-3 = (2a+3)(2b-1)$, $n = 2|(-a)b| + (-a) + 3b$

Therefore, all cases have a form of $2|ab| + a + 3b$. This means if there are infinitely many integers n that does not has the form of $2|ab| + a + 3b$ then there are infinitely many twin prime pairs $(2n-3, 2n+3)$.

Next, let us consider the form of integer n when there are infinitely many twin prime pairs $(2n-3, 2n+3)$. If there is a prime pair $(2n-3, 2n+3)$, which n has a form of $2|ab| + a + 3b$ then there are four cases as above. The four cases all contains at least one composite number in the pair, which is contradiction. Therefore, if there are infinitely many twin prime pairs $(2n-3, 2n+3)$ then there are infinitely many integers n that cannot be expressed in the form of $2|ab| + a + 3b$. $\square$

THEOREM 2.6. *It is equivalent that there are infinitely many integers n that cannot be expressed as $2\,|ab| + a + kb$ where a, b are integers, k is an odd with $a, b \neq \pm\frac{k+1}{2}, \pm\frac{k-1}{2}$, $b \neq 0, \pm 1$, and there are infinitely many general twin prime pairs with the difference $2k$ and having the form $(2n-k, 2n+k)$.*

Proof. It is proved by dividing into the case of prime number and composite number.

1. The case of k is a prime number :
   All odd composite number should be expressed on the form of
$$(2a+p)(2b+1), (2a-p)(2b-1), (2a-p)(2b+1), (2a+p)(2b-1).$$
   Similarly like the proof of the theorem 2.5, we have desired result.
2. The case of k is a composite number :
   Let $k = \alpha\beta$ where $\alpha, \beta \neq 1$, and α, β are odd numbers. All odd composite number should be expressed on the form of
$$(2a+\alpha)(2b+\beta), (2a-\alpha)(2b-\beta), (2a-\alpha)(2b+\beta), (2a+\alpha)(2b-\beta).$$

If we replace a as $a' + \frac{\alpha\beta-\alpha}{2}$ and b as $b' + \frac{1-\beta}{2}$ then we get the form of $n = 2|ab| + a + \alpha\beta b = 2|ab| + a + kb$.

$\square$

References

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